$\partial x \quad \partial y$

Q.2. If $u = f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. *(2017)*

Sol. We have, $$\frac{\partial u}{\partial x} = \left[f'\left(\frac{-y}{x}\right)\right]\left(\frac{-y}{x^2}\right)$$

(Diff. partially w.r.t. x)

$$\therefore \quad x\frac{\partial u}{\partial x} = -\left(\frac{y}{x}\right)f'\left(\frac{y}{x}\right) \qquad ...(i)$$

Again, $$\frac{\partial u}{\partial y} = \left[f'\left(\frac{y}{x}\right)\right].\left(\frac{1}{x}\right)$$

(Diff. partially w.r.t. y)

$$\therefore \quad y\left(\frac{\partial u}{\partial y}\right) = \left(\frac{y}{x}\right)f'\left(\frac{y}{x}\right) \qquad ...(ii)$$

Adding Eqs. (i) and (ii), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Proved.

Q.3. If $z = \tan(y + ax) + (y - ax)^{3/2}$, then find the value of $(\partial^2 z / \partial x^2) - a^2(\partial^2 z / \partial y^2)$.

Sol. Given, $z = \tan(y + ax) + (y - ax)^{3/2}$

$$\therefore \quad \left(\frac{\partial z}{\partial x}\right) = \{\sec^2(y + ax)\} \cdot a + \frac{3}{2}(y - ax)^{1/2}.(-a)$$

$$\text{and} \left(\frac{\partial^2 z}{\partial x^2}\right) = 2a^2 \tan(y + ax)\sec^2(y + ax) + \frac{3}{4}a^2(y - ax)^{-1/2}$$

$$\text{Again,} \left(\frac{\partial z}{\partial y}\right) = \sec^2(y + ax) + \frac{3}{2}(y - ax)^{1/2}$$

$$\text{and} \left(\frac{\partial^2 z}{\partial x^2}\right) = 2\sec^2(y + ax)\tan(y + ax) + \frac{3}{4}(y - ax)^{-1/2}$$

$$\text{Thus,} \left(\frac{\partial^2 z}{\partial x^2}\right) - a^2\left(\frac{\partial^2 z}{\partial y^2}\right) = 0$$

Ans.

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Q.4. If $u = e^{xyz}$, then show that $\partial^3 u / \partial x\,\partial y\,\partial z = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

Sol. Given, $u = e^{xyz}$

$\therefore \quad \partial u/\partial z = xye^{xyz}$

Now, $$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}(xye^{xyz}) = x\frac{\partial}{\partial y}(ye^{xyz})$$

$$= x[y \cdot xze^{xyz} + e^{xyz}] = e^{xyz}(x^2yz + x)$$

Again, $$\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\,\partial z}\right) = \frac{\partial}{\partial x}[e^{xyz}(x^2yz + x)]$$

$$= e^{xyz}(2xzy + 1) + yze^{xyz}(x^2yz + x)$$

$$= e^{xyz}(2xyz + 1 + x^2y^2z^2 + xyz)$$

$$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$$ **Proved.**

Q.5. Find $\frac{\partial f}{\partial x}$, if $f = ye^{(x^2+y^2)}$. *(2016)*

Sol. Given, $f = ye^{(x^2+y^2)}$

Taking log on both sides, $\log f = \log y + (x^2 + y^2)$

$$\frac{1}{f}\frac{\partial f}{\partial x} = 0 + 2x + 0$$

$$\frac{\partial f}{\partial x} = 2x \cdot ye^{(x^2+y^2)}$$ **Ans.**

Q.6. Find $\frac{\partial f}{\partial n}$, if $f = \log(x^2 + y^2)$.

Sol. Given, $f = \log(x^2 + y^2)$

$$|\nabla f| = \frac{\partial f}{\partial n}$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} = i\frac{2x}{(x^2 + y^2)} + j\frac{2y}{x^2 + y^2}$$

$$|\nabla f| = \sqrt{\frac{4x^2}{(x^2 + y^2)^2} + \frac{4y^2}{(x^2 + y^2)^2}} = \frac{2}{\sqrt{x^2 + y^2}}.$$ **Ans.**

Q.7. If $x^y + y^x = a^b$, then find dy/dx.

Sol. Let $f(x, y) = x^y + y^x - a^b$

Then, we have $f(x, y) = 0$

$\therefore$ $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$ **Ans.**

Q.8. Show that the differential equation $d^2x/dy^2 = a$ may be written in the form $(d^2y/dx^2) + a(dy/dx)^3 = 0$.

Sol. We have, $$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$$ ...(i)

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$$\therefore \quad \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left\{\left(\frac{dy}{dx}\right)^{-1}\right\} \qquad \text{[From Eq. (i)]}$$

$$= \frac{d}{dx}\left\{\left(\frac{dy}{dx}\right)^{-1}\right\}\cdot\frac{dx}{dy} = -\left(\frac{dy}{dx}\right)^{-2}\cdot\frac{d^2y}{dx^2}\cdot\left(\frac{dy}{dx}\right)^{-1}$$

$$= -\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}$$

Hence, the differential equation $d^2x/dy^2 = a$ becomes

$$-\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2} = a \quad \text{or} \quad \frac{d^2y}{dx^2} = -a\left(\frac{dy}{dx}\right)^3$$

or $(d^2y/dx^2) + a(dy/dx)^3 = 0$ **Proved.**

Q.9. Find the value of $\frac{\partial^2 z}{\partial x \partial y}$, when $z = \sin^{-1}\left(\frac{x}{y}\right)$.

Sol. Given, $z = \sin^{-1}\left(\frac{x}{y}\right)$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-(x/y)^2}}\cdot\frac{1}{y} = \frac{1}{\sqrt{y^2-x^2}} = (y^2-x^2)^{-1/2}$$

Now, $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}[(y^2-x^2)^{-1/2}] = -\frac{1}{2}(y^2-x^2)^{-3/2}.2y$$

$$= \frac{-y}{(y^2-x^2)^{3/2}}$$ **Ans.**

Q.10. If $u = x^2y^2/(x+y)$, then show that $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 3u$.

Sol. Given, $u = \frac{x^2y^2}{x+y} = \frac{x^3(y/x)^2}{[1+(y/x)]} = x^3 f(y/x)$, (say)

Thus, u is a homogeneous function of x and y of degree 3.

Therefore, by Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$$ **Proved.**

Q.11. Define maximum value.

Ans. A function $f(x,y)$ is said to have a maximum value at $x = a, y = b$, if there exists a small neighbourhood of (a,b) such that

$$f(a,b) > f(a+h, b+k).$$

Q.12. What is extreme or extremum value?

Ans. The maximum and minimum values of a function are also called extreme or extremum values of the function.

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Section-B (Short Answer Questions)

Q.1. If u be a homogeneous function of degree n, show that

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\,\partial y}=(n-1)\frac{\partial u}{\partial x}.$$

Sol. Let u be a homogeneous function of degree n, then by Euler's theorem.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Partially differentiating w.r.t.x, we get

$$\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)+y\frac{\partial^2 u}{\partial x\,\partial y}=n\frac{\partial u}{\partial x}.$$

$$\left(x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\right)+y\frac{\partial^2 u}{\partial x\,\partial y}=n\frac{\partial u}{\partial x}.$$

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\,\partial y}=(n-1)\frac{\partial u}{\partial x}$$

Proved.

Q.2. If $u=\tan^{-1}\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$. *(2018)*

Sol. We have, $u=\tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \tan u=\frac{y}{x}$

Let $f(x,y)=\frac{y}{x}=\tan u$

f is a homogeneous function of degree 0 in x, y.

Thus, by Euler's theorem on homogeneous functions,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0 \quad \text{...(i)}$$

Now, $f(x,y)=\tan u$

$\therefore \quad \frac{\partial f}{\partial x}=\sec^2 u\frac{\partial u}{\partial x}$ and $\frac{\partial f}{\partial y}=\sec^2 u\frac{\partial u}{\partial y}$

Putting these values in Eq. (i), we get

$$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=0$$

$$\Rightarrow \quad \sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=0$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Proved.

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Q.3. If $u = \tan^{-1}\dfrac{xy}{\sqrt{(1+x^2+y^2)}}$, then show that $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{1}{(1+x^2+y^2)^{3/2}}$.

Sol. Given,

$$u = \tan^{-1}\frac{xy}{\sqrt{(1+x^2+y^2)}}$$

Differentiating partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = \frac{1}{1+\dfrac{x^2y^2}{1+x^2+y^2}} \cdot y \cdot \frac{1\cdot\sqrt{(1+x^2+y^2)} - x\cdot\dfrac{1}{2}(1+x^2+y^2)^{-1/2}(-2x)}{1+x^2+y^2}$$

$$= \frac{1+x^2+y^2}{1+x^2+y^2+x^2y^2} \cdot y \cdot \frac{(1+x^2+y^2)-x^2}{(1+x^2+y^2)(1+x^2+y^2)^{1/2}}$$

$$= \frac{y(1+y^2)}{(1+x^2)(1+y^2)(1+x^2+y^2)^{1/2}}$$

$$= \frac{1}{1+x^2}\cdot\frac{y}{(1+x^2+y^2)^{1/2}}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{1}{1+x^2}\cdot\frac{1\cdot(1+x^2+y^2)^{1/2} - y\cdot\dfrac{1}{2}(1+x^2+y^2)^{-1/2}\cdot 2y}{1+x^2+y^2}$$

$$= \frac{1}{1+x^2}\cdot\frac{1+x^2+y^2-y^2}{(1+x^2+y^2)(1+x^2+y^2)^{1/2}}$$

$$= \frac{1}{1+x^2}\cdot\frac{1+x^2}{(1+x^2+y^2)^{3/2}}$$

$$= \frac{1}{(1+x^2+y^2)^{3/2}}$$

Proved.

Q.4. If $u = \sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$, then show that $x\left(\dfrac{\partial u}{\partial x}\right) + y\left(\dfrac{\partial u}{\partial y}\right) = \tan u$. ***(2016)***

Sol. Given, $\sin u = (x^2+y^2)/(x+y)$

$$\log\sin u = \log(x^2+y^2) - \log(x+y) \qquad \text{...(i)}$$

Differentiating Eq. (i) partially w.r.t x, we get

$$\frac{1}{\sin u}\cos u\cdot\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x+y}$$

$$\therefore \quad (\cot u)x\frac{\partial u}{\partial x} = \frac{2x^2}{x^2+y^2} - \frac{x}{x+y} \qquad \text{...(ii)}$$

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Again differentiating Eq. (i) partially w.r.t.y, we get

$$\frac{\cos u}{\sin u}\cdot\frac{\partial u}{\partial y}=\frac{2y}{x^2+y^2}-\frac{1}{x+y}$$

$$\therefore\quad (\cot u)\cdot y\frac{\partial u}{\partial y}=\frac{2y^2}{x^2+y^2}-\frac{y}{x+y}$$

Adding Eqs. (ii) and (iii), we get

$$(\cot u)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\frac{2x^2+2y^2}{x^2+y^2}-\frac{x+y}{x+y}=2-1=1$$

$$\therefore\quad x\left(\frac{\partial u}{\partial x}\right)+y\left(\frac{\partial u}{\partial y}\right)=\left(\frac{1}{\cot u}\right)=\tan u$$

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