

Proved.

Thus the two integrals are not equal.

Q.9. Find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$, by double integration.

Sol. From the equation of the ellipse, we have

$$\frac{y}{b} = \pm \sqrt{\left\{1 - \frac{x^2}{a^2}\right\}}$$

So the region of integration R to cover the area of the ellipse can be considered by bounded by

$$y = -b\sqrt{(1 - x^2/a^2)}, y = b\sqrt{(1 - x^2/a^2)}, x = -a \text{ and } x = a$$

Therefore, the required area of the ellipse

$$= \iint_R dx dy = \int_{x=-a}^a \int_{y=-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}} 1. dx dy$$

$$\begin{aligned}
&= \int_{-a}^a \left[2 \int_0^{b\sqrt{(1-x^2/a^2)}} 1 dy \right] dx = 2 \int_{-a}^a [y]_0^{b\sqrt{(1-x^2/a^2)}} dx \\
&= 2 \int_{-a}^a b \sqrt{\left(1 - \frac{x^2}{a^2}\right)} dx = 2.2 \int_0^a b \sqrt{\left(1 - \frac{x^2}{a^2}\right)} dx \\
&= \frac{4b}{a} \int_0^a \sqrt{(a^2 - x^2)} dx = \frac{4b}{a} \left[\frac{x\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a \\
&= \frac{4b}{a} \left[0 + \frac{a^2}{2} \{\sin^{-1} 1 - \sin^{-1} 0\} \right] = \frac{4b}{a} \cdot \frac{a^2}{2} \cdot \frac{\pi}{2} = \pi ab
\end{aligned}$$

Ans.

Q.10. Evaluate $\iint (x^2 + y^2) dx dy$ over the region in the positive quadrant for which $x + y \leq 1$ (2014)

Sol. The region of integration R is the area bounded by the coordinate axes and the straight line $x + y = 1$. Therefore, the region R is bounded by $y = 0$, $y = 1 - x$ and $x = 0$, $x = 1$.

Therefore,

$$= \iint_R (x^2 + y^2) dx dy = \int_{x=0}^1 \int_{y=0}^{1-x} (x^2 + y^2) dx dy$$

(Integrating w.r.t. y regarding x as a constant)

$$\begin{aligned}
&= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx = \int_0^1 \left[x^2 (1-x) + \frac{(1-x)^3}{3} \right] dx \\
&= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{3 \times 4} \right]_0^1 = \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{12} \right] = \frac{1}{6}
\end{aligned}$$

Ans.

Q.11. Evaluate $\iint xy dx dy$ over the region in the positive quadrant for which $x + y \leq 1$.

Sol. Here the region can be expressed as $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$.

$$\therefore \iint xy dx dy = \int_0^1 \int_0^{1-x} xy dx dy$$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_0^{1-x} dx = \frac{1}{2} \int_0^1 x (1-x)^2 dx$$

$$= \frac{1}{2} \int_0^1 x(1+x^2 - 2x) dx = \frac{1}{2} \int_0^1 (x^3 - 2x^2 + x) dx$$

$$= \frac{1}{2} \left[\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} \right]_0^1 = \frac{1}{2} \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right)$$

$$= \frac{1}{2} \times \frac{(3-8+6)}{12} = \frac{1}{24}$$

Ans.

Q.12. Evaluate $\iint e^{2x+3y} dx dy$ over the triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$.

Sol. The given region of integration R can be expressed as
 $0 \leq x \leq 1, 0 \leq y \leq 1-x$

where the first integration is to be performed w.r.t. y treating x as a constant.
 $\therefore \iint_R e^{2x+3y} dx dy = \int_0^1 \int_0^{1-x} e^{2x+3y} dx dy$

$$\begin{aligned}
 &= \int_0^1 \left[\frac{e^{2x+3y}}{3} \right]_0^{1-x} dx = \frac{1}{3} \int_0^1 [e^{3-x} - e^{2x}] dx \\
 &= \frac{1}{3} \left[-e^{3-x} - \frac{e^{2x}}{2} \right]_0^1 = -\frac{1}{3} [(e^2 - e^3) + \frac{1}{2}(e^2 - e^0)] \\
 &= -\frac{1}{3} \left[-e^2(e-1) + \frac{1}{2}(e+1)(e-1) \right] = \frac{1}{3}(e-1) \left[e^2 - \frac{1}{2}(e+1) \right] \\
 &= \frac{1}{6}(e-1)(2e^2 - e - 1) = \frac{1}{6}(e-1)(e-1)(2e+1) \\
 &= \frac{1}{6}(e-1)^2(2e+1)
 \end{aligned}$$

Ans.

Q.13. Evaluate $\iint xy(x+y) dx dy$ over the area between $y = x^2$ and $y = x$.

Sol. Draw the given curves $y = x^2$ and $y = x$ in the same figure. The two curves intersect at the points whose abscissae are given by $x^2 = x$ or $x(x-1) = 0$, i.e. $x = 0$ or 1. When $0 < x < 1$, we have $x > x^2$, so the area of integration can be considered as lying between the curve $y = x^2$, $y = x$, $x = 0$ and $x = 1$.

Therefore, the required integral

$$\begin{aligned}
 &= \int_{x=0}^1 \int_{y=x^2}^x xy(x+y) dx dy = \int_0^1 \left[\int_{x^2}^x (x^2 y + xy^2) dy \right] dx \\
 &= \int_0^1 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^x dx = \int_0^1 \left[\left(\frac{x^4}{2} + \frac{x^4}{3} \right) - \left(\frac{x^6}{2} + \frac{x^7}{3} \right) \right] dx \\
 &= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right] dx = \left[\frac{x^5}{6} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1 \\
 &= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{28-12-7}{168} = \frac{9}{168} = \frac{3}{56}
 \end{aligned}$$

Ans.

Q.14. Find the area between the line $y = x$ and curve $y = x^2$ enclosed in first quadrant. (2016)

Sol. Solving eqs. $y = x^2$ and $y = x$, we get

$$x^2 - x = 0$$

$$x(x-1) = 0$$

or

130

i.e.

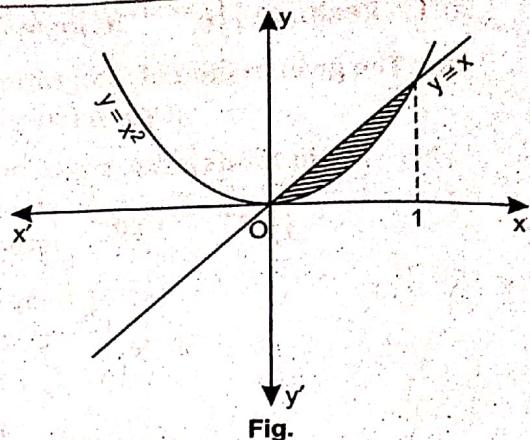
$$x = 0, 1$$

Thus, the curves $y = x^2$ and $y = x$ intersect at the points where, $x = 0$ and $x = 1$.

when $0 < x < 1$,
we have, $x > x^2$

So, the required area lies between the curves $y = x$, $y = x^2$, $x = 0$ and $x = 1$

$$\begin{aligned}\therefore \text{The required area} &= \int_{x=0}^1 \int_{y=x^2}^x dx dy \\ &= \int_0^1 [y]_{x^2}^x dx = \int_0^1 (x - x^2) dx \\ &= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\end{aligned}$$



Ans.

Q.15. Prove by the method of double integration that the area lying between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Sol. Draw the two parabolas in the same figure. The two parabolas intersect at the points whose abscissae are given by $(x^2/4a)^2 = 4ax$, i.e. $x(x^3 - 64a^3) = 0$, i.e. $x = 0$ and $x^3 = 64a^3$. Thus, the two parabolas intersect at the points where $x = 0$ and $x = 4a$.

Now, the area of a small element situated at any point $(x, y) = dx dy$

\therefore The required area

$$\begin{aligned}&= \int_{x=0}^{4a} \int_{y=x^2/4a}^{\sqrt{(4ax)}} dx dy = \int_0^{4a} [y]_{x^2/4a}^{\sqrt{(4ax)}} dx \\ &= \int_0^{4a} \left[2\sqrt{a} \cdot x^{1/2} - \frac{1}{4a} \cdot x^2 \right] dx = \left[2\sqrt{a} \cdot x^{3/2} \cdot \frac{2}{3} - \frac{1}{4a} \cdot \frac{x^3}{3} \right]_0^{4a} \\ &= \frac{4}{3} \sqrt{a} \cdot (4a)^{3/2} - \frac{1}{12a} \cdot 64a^3 = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2\end{aligned}$$

Proved.

Q.16. Evaluate $\iint \frac{xy}{\sqrt{(1-y^2)}} dx dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.

Sol. Here, the region of integration R is the area of the circle $x^2 + y^2 = 1$ lying in the positive quadrant. This region of integration R can be expressed as

$$0 \leq x \leq \sqrt{(1-y^2)}, 0 \leq y \leq 1$$

$$\begin{aligned}\therefore \iint_R \frac{xy}{\sqrt{(1-y^2)}} dx dy &= \int_{y=0}^1 \int_{x=0}^{\sqrt{(1-y^2)}} \frac{xy}{\sqrt{(1-y^2)}} dx dy \\ &= \int_0^1 \frac{y}{\sqrt{(1-y^2)}} \left[\frac{x^2}{2} \right]_{x=0}^{\sqrt{(1-y^2)}} dy\end{aligned}$$

$$\begin{aligned}&\quad \text{(Integrating w.r.t. } x \text{ treating } y \text{ as a constant)} \\ &= \frac{1}{2} \int_0^1 y \sqrt{(1-y^2)} dy = \frac{1}{2} \int_0^1 \frac{1}{2} \cdot (1-y^2)^{1/2} (-2y) dy\end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{4} \cdot \frac{2}{3} [(1-y^2)^{3/2}]_0^1 \quad [\text{By power formula}] \\
 &= \frac{1}{6}
 \end{aligned}$$

Ans.

Q.17. When the region of integration A is the triangle given by $y = 0$, $y = x$ and $x = 1$, show that

$$\iint_A \sqrt{(4x^2 - y^2)} dx dy = \frac{1}{3} \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right).$$

Sol. In the diagram, draw the straight lines $y = 0$, $y = x$ and $x = 1$. Then, we observe that region of integration A can be expressed as

$$\begin{aligned}
 &\because 0 \leq y \leq x, 0 \leq x \leq 1. \\
 \iint_A \sqrt{(4x^2 - y^2)} dx dy &= \int_{x=0}^1 \int_{y=0}^x \sqrt{(4x^2 - y^2)} dx dy \\
 &= \int_0^1 \left[\frac{y}{2} \sqrt{(4x^2 - y^2)} + 2x^2 \sin^{-1} \frac{y}{2x} \right]_{y=0}^x dx \\
 &\quad (\text{Integrating w.r.t. } y \text{ treating } x \text{ as a constant}) \\
 &= \int_0^1 \left[\frac{x}{2} \sqrt{(4x^2 - x^2)} + 2x^2 \sin^{-1} \frac{1}{2} - 0 \right] dx \\
 \int_0^1 \left[\frac{\sqrt{3}}{2} x^2 + \frac{\pi}{3} x^2 \right] dx &= \left[\frac{\sqrt{3}}{2} \cdot \frac{x^3}{3} + \frac{\pi}{3} \cdot \frac{x^3}{3} \right]_0^1 \\
 &= \frac{1}{3} \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right)
 \end{aligned}$$

Proved.

Q.18. Evaluate the double integral $\int_0^a \int_{y=0}^{\sqrt{(a^2-x^2)}} x^2 y dx dy$.

Mention the region of integration involved in this double integral.
Sol. Let,

$$\begin{aligned}
 I &= \int_{x=0}^a \int_{y=0}^{\sqrt{(a^2-x^2)}} x^2 y dx dy \\
 &= \int_0^a x^2 \left[\frac{y^2}{2} \right]_{y=0}^{\sqrt{(a^2-x^2)}} dx
 \end{aligned}$$

(Integrating w.r.t. y treating x as a constant)

$$\begin{aligned}
 &= \frac{1}{2} \int_0^a x^2 (a^2 - x^2) dx = \frac{1}{2} \int_0^a (a^2 x^2 - x^4) dx \\
 &= \frac{1}{2} \left[a^2 \frac{x^3}{3} - \frac{x^5}{5} \right]_0^a = \frac{1}{2} \left[\frac{a^5}{3} - \frac{a^5}{5} \right] = \frac{1}{15} a^5
 \end{aligned}$$

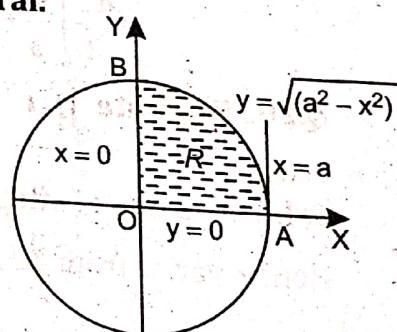


Fig.

From the limits of integration, it is obvious that the region of integration R is bounded by $y = 0$, $y = \sqrt{(a^2 - x^2)}$ and $x = 0$, $x = a$, i.e. the region of integration is the area of the circle $x^2 + y^2 = a^2$ between the lines $x = 0$, $x = a$ and lying above the line $y = 0$, i.e. the axis of x. Thus, the region of integration is the area OAB of the circle $x^2 + y^2 = a^2$ lying in the positive quadrant.

Ans.