Q.16. Define composite function with suitable example.

Or **Let $X = \{1, 2, 3\}$ and f and g be functions from X to itself given by $f = \{(1, 2), (2, 3), (3, 1)\}$ and $g = \{(1, 1), (2, 2), (3, 1)\}$. Find fog and gof.** *(2016)*

Ans. Composite Function : Let f be a function from X to Y and let g be a function from Y to Z. Then, the composite of the function f and g denoted by gof is a mapping from X to Z defined by,

$$(gof)(x) = g\cdot[f(x)]\ \forall\ x \in X$$

For example, Given, $f : X \to X$ and $g : X \to X$

$\therefore$ $gof : X \to X$ defined by

$$(gof)(x) = g(f(x))$$

$\therefore$
$$(gof)(1) = g(f(1)) = g(2) = 2$$
$$(gof)(2) = g(f(2)) = g(3) = 1$$
$$(gof)(3) = g(f(3)) = g(1) = 1$$

$\therefore$ $gof = \{(1, 2), (2, 1), (3, 1)\}$ **Ans.**

Similarly, $fog = \{(1, 2), (2, 3), (3, 2)\}$ **Ans.**

We see that $gof \neq fog$

Q.17. Prove that the composition of any function with the identity function is the function itself.

Sol. Let $f : A \to B$ be a function and $I_A : A \to A$ be an identity function.

Since, $I_A : A \to A$ and $f : A \to B$, therefore $f o I_A : A \to B$.

Let $x \in A$, then $(f o I_A)(x) = f(I_A(x)) = f(x)$ (By definition of identity function)

$$I_A(x) = x, \forall\ x \in (A)$$

$\Rightarrow$ $f o I_A = f$

Also, $f : A \to B$ and $I_B : B \to B$ $\Rightarrow$ $I_B o f : A \to B$

Let $x \in B$ and let $f(x) = y$, then $y \in B$.

Therefore, $(I_B o f)(x) = I_B(f(x)) = I_B(y) = y = f(x)$

Therefore, $I_B o f = f$ **Proved.**

Hence, $f o I_A = I_B o f = f$

Q.18. If $f : X \to Y$ and $g : Y \to Z$ are bijections, show that $(g o f)^{-1} = f^{-1} o g^{-1}$. *(2015)*

Sol. Since f and g are one-one and onto, therefore they are bijective. Hence, $g\ o\ f : X \to Z$ is also bijective.

Therefore, $g o f$ is also invertible. ...(i)

Let $x \in X$, then $\exists\ y \in Y$, such that $y = f(x)$ ...(ii)

and let $z \in Z$, then $\exists\ y \in Y$, such that $z = g(y)$.

$$(gof)(x) = g[f(x)] = g(y) = z \quad \text{...(iii)}$$
$$(gof)^{-1}(z) = x$$

From Eqs. (i) and (ii), we get

$$x = f^{-1}(y) \quad \text{and} \quad y = g^{-1}(z)$$

$$(f^{-1} o g^{-1})(z) = f^{-1}[g^{-1}(z)] = f^{1}(y) = x \quad \text{...(iv)}$$

$$(f^{-1} o g^{-1})z = x$$

From Eqs. (iii) and (iv), we get

$$(f^{-1}og^{-1}) = (gof)^{-1}$$ **Proved.**

Q.19. Let $f: R \to R$ and $g: R \to R$ be defined by $f(x) = x - 1$ and $g(x) = x^2 + 1$. Find $fog(2)$, $gof(2)$, $fof(2)$ and $gog(2)$. **(2017)**

Sol. Since, we have $f(x) = x - 1$ and $g(x) = x^2 + 1$

(i) $fog(x) = f[g(x)] = f(x^2 + 1) = x^2 + 1 - 1 = x^2$

$\therefore \quad fog(2) = 2^2 = 4 \Rightarrow fog(2) = 4$ **Ans.**

(ii) $gof(x) = g[f(x)] = g(x-1) = (x-1)^2 + 1 = x^2 + 2 - 2x$

$\therefore \quad gof(2) = (2)^2 + 2 - 2(2)$

$= 4 + 2 - 4 = 2 \Rightarrow gof(2) = 2$ **Ans.**

(iii) $fof(x) = f[f(x)] = f(x-1) = (x-1) - 1 = x - 2$

$\therefore \quad fof(2) = 2 - 2 = 0 \Rightarrow fof(2) = 0$ **Ans.**

(iv) $gog(x) = g[g(x)] = g(x^2 + 1)$

$= (x^2+1)^2 + 1$

$= x^4 + 1 + 2x^2 + 1$

$= x^4 + 2x^2 + 2$

$\therefore \quad gog(2) = (2)^4 + 2(2)^2 + 2$

$= 16 + 8 + 2 = 26$

or $\quad gog(2) = 26$ **Ans.**

Q.20. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then show that $f(x) + f(y) = f\left(\frac{x+y}{1+xy}\right)$. **(2018)**

Sol. Given,

$$f(x) = \log\left(\frac{1+x}{1-x}\right) \quad \text{...(i)}$$

and

$$f(y) = \log\left(\frac{1+y}{1-y}\right) \quad \text{...(ii)}$$

Adding Eqs. (i) and (ii), we get

$$f(x) + f(y) = \log\left(\frac{1+x}{1-x}\right) + \log\left(\frac{1+y}{1-y}\right)$$

$$= \log\left[\frac{(1+x)(1+y)}{(1-x)(1-y)}\right] = \log\left[\frac{1+x+y+xy}{1-x-y+xy}\right] \quad \text{...(iii)}$$

Again,

$$f\left(\frac{x+y}{1+xy}\right) = \log\left[\frac{1+\frac{x+y}{1+xy}}{1-\frac{x+y}{1+xy}}\right]$$

$$= \log\left[\frac{1+x+y+xy}{1-x-y+xy}\right] \quad \text{...(iv)}$$

Using Eqs. (iii) and (iv), we conclude that

$$f(x) + f(y) = f\left(\frac{x+y}{1+xy}\right)$$

Proved.

Q.21. If $f(x) = x^2 - \frac{1}{x^2}$, show that $f(x) + f\left(\frac{1}{x}\right) = 0$.

Sol. We have, $f(x) = x^2 - \frac{1}{x^2}$

Now, we have to prove that $f(x) + f\left(\frac{1}{x}\right) = 0$

$$f(x) = x^2 - \frac{1}{x^2}$$

and $$f\left(\frac{1}{x}\right) = \frac{1}{x^2} - x^2$$

Therefore, $$f(x) + f\left(\frac{1}{x}\right) = x^2 - \frac{1}{x^2} + \frac{1}{x^2} - x^2$$

$$= 0$$

Proved.